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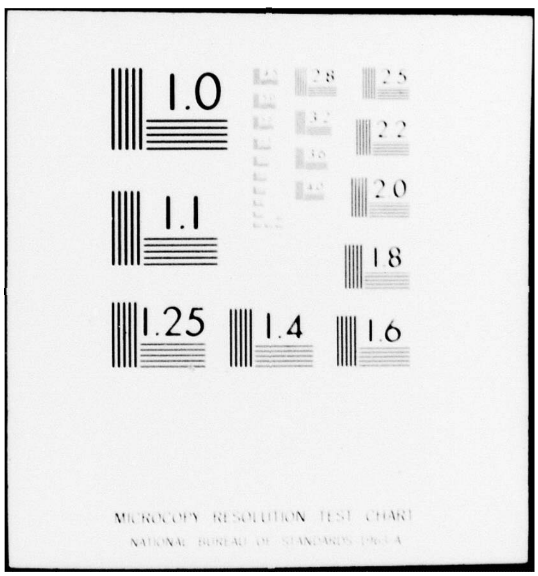
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SINGULAR PERTURBATIONS AND SLOW-MODE
APPROXIMATIONS FOR LARGE SCALE LINEAR SYSTEMS

by

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Abstract

The solutions of large linear systems with slow and fast modes are well approximated away from endpoints by a slow-mode solution. This corresponds to the use of outer solutions as approximate solutions of singular perturbation problems away from boundary layer regions. This paper gives an algorithm for obtaining the slow-mode solutions, and illustrates how the slow and fast components are obtained for the initial value problem.

Decoupling of slow and fast modes and the singular perturbations methodology are alternative ways of reducing the order of high dimensional mathematical models in control and stability theory. Physically, one sometimes knows which variables involve predominantly fast or slow motions, although complete slow-fast decoupling is rarely known. Likewise, system models are seldom presented in the traditional singularly perturbed form $\dot{u} = f(u,v,t,\varepsilon)$, $\varepsilon\dot{v} = g(u,v,t,\varepsilon)$ to which standard theories can be directly applied. Instead, a modeler must numerically identify the small parameters involved before he can

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effectively use the singular perturbations structure to design reduced-order models. One can attempt to transform the problem into an equivalent problem with decoupled slow and fast modes, but care must then be used to maintain the physical interpretations for the variables used. Despite all difficulties, the high dimensionality of truly descriptive models prohibits quick numerical integration and forces one to approximate solutions via lower-order models corresponding to a slow-mode approximation or an outer solution to a traditional singular perturbation problem.

The simplest example of such a situation is provided by the linear constant coefficient system

$$\dot{x} = Ax \tag{1}$$

on a finite interval, say $0 \le t \le 1$, where the eigenvalues of A lie in two sets whose elements differ greatly in magnitude. Such systems could, of course, come from a control problem with state equation $\dot{x} = \tilde{A}x + \tilde{B}u$, feedback control $u = Gx$, and $A = \tilde{A} + \tilde{B}G$, especially when G provides high-gain feedback (cf. Young et al. (1977)). Specifically, *let us decompose the spectrum* $\lambda(A)$ *of* A into a slow set S and *a fast (but not purely oscillatory) set* F with n_1 and n_2 *elements, respectively, such that* $n_1 + n_2 = n$. Thus, we'll take

$$\lambda(A) = S \cup F \tag{2}$$

where

$$|s| \ll \operatorname{Re} |f| \tag{3}$$

whenever $s \in S$ and $f \in F$. (If we asked only that $|s| \ll |f|$, we would also allow large, but (nearly) purely imaginary, eigenvalues leading to rapidly oscillating modes. Instead of the boundary layer analysis that we shall employ, the quite-different method of averaging would have to be utilized (cf. Hoppensteadt and Miranker (1976)).) We note that the partitioning (2) of $\lambda(A)$ introduces a small, dimensionless parameter

$$\epsilon = \max_{s \in S} |s| / \min_{f \in F} |f| \ll 1. \tag{4}$$

If a more refined partitioning of the spectrum were used, we would obtain several small parameters of decreasing size, corresponding to modes with several distinct decay rates.

Our algorithm for obtaining approximate slow-mode solutions of (1) will be

as follows:

(a) obtain the eigenvalues of the matrix A and select a set S of slow eigenvalues satisfying (3),

(b) find an $n_1 \times n$ matrix solution $M_s = \begin{bmatrix} M_{s1} \\ M_{s2} \end{bmatrix}$ of the equation

$$AM_s = M_s J_s \tag{5}$$

where the $n_1 \times n_1$ matrix J_s has the prescribed spectrum $\lambda(J_s) = S$. Rearranging the rows of x, if necessary, so that M_{s1} is invertible, define

$$L = -M_{s2}M_{s1}^{-1}. \tag{6}$$

(c) The slow-mode solutions to (1) lie in the column span of the exact $n_1 \times n$ matrix solution

$$X_s(t) = \begin{bmatrix} I_{n_1} \\ L \end{bmatrix} e^{(A_{11}-A_{12}L)t}, \tag{7}$$

where we have used the partitioning $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$.

Remarks:

(i) Different workers might naturally make different choices of the number n_1 of slow eigenvalues. Our experience suggests that the approximations to solutions of boundary value problems with bounded solutions will remain relatively good within $(0,1)$ even for moderately large ε values (say $\varepsilon \approx 0.4$). The primary advantages are gained when n_1/n is small.

(ii) We could take J_s to be the Jordan form corresponding to the slow eigenvalues of A (though this might be imprudent from a numerical viewpoint), with M_s spanning the corresponding n_1 dimensional generalized eigenspace. Rearranging the rows of M_s, then, we can obtain a nonsingular matrix M_{s1} and thereby a solution L.

(iii) It is particularly convenient numerically to obtain a matrix L of small norm. This will correspond to having the last n_2 entries of x be primarily fast. Such a condition can often be achieved by scaling and balancing techniques which reduce the coupling between slow and fast modes. Alternatively, a decoupling matrix L with small norm would result if physical coordinates known to be primarily slow were used as the first n_1 components of x while coordinates dominated by fast variation be used for the final components.

In general, L is not necessarily small. Instead it is characterized as the $n_1 \times n_2$ dimensional matrix such that the change of variables

$$z = \begin{pmatrix} I_{n_1} & 0 \\ L & I_{n_2} \end{pmatrix} x$$

provides the block-triangular system

$$\dot{z} = \begin{pmatrix} B_1 & A_{12} \\ 0 & B_2 \end{pmatrix} z$$

where

(8) $$\lambda(B_1) = S \quad \text{for} \quad B_1 = A_{11} - A_{12}L,$$

(9) $$\lambda(B_2) = F \quad \text{for} \quad B_2 = A_{22} + LA_{12},$$

and where L satisfies the algebraic Riccati equation

(10) $$LA_{11} - A_{22}L - LA_{12}L + A_{21} = 0.$$

Alternative ways of calculating L involve the iterative solution of this quadratic equation for a sufficiently good initial guess. For small L and A_{22} nonsingular, we have $L \approx A_{22}^{-1}A_{21}$, justifying the initial iterate used in such instances by Chow and Kokotovic (1976). Once n_1 is chosen and A is fixed, there is a unique real matrix L satisfying (8)-(10).

(iv) Because A generally has both large stable and unstable eigenvalues, the $n \times n$ fundamental matrix e^{At} for (1) cannot be readily computed. Since $B_1 = A_{11} - A_{12}L$ has only the moderate eigenvalues of S, however, the $n_1 \times n_1$ matrix $e^{B_1 t}$ can be obtained by explicit formulas or by many standard methods of numerical integration.

If F contains only (large) stable eigenvalues, the solution of the initial value problem for (1) will be bounded and for $t > 0$ it will be well approximated by $X_s k$ for some vector k. Solutions to terminal value problems for (1) would then become asymptotically large (as $\epsilon \to 0$) away from the final endpoint, unless the terminal values were restricted to an n_1 dimensional manifold on which the fast growing modes would not be excited. In general, when F contains m_1 eigenvalues with negative real parts and $m_2 = n_2 - m_1$ eigenvalues with positive real parts, (1) will have an m_1 dimensional manifold of boundary layer solutions which decay to zero away from the initial endpoint and an m_2 dimensional manifold of solutions which decay to zero away from the terminal endpoint (alternatively, an m_2 dimensional manifold of rapidly growing solutions

near $t = 0$). Boundary value problems must be restricted to those whose solutions remain bounded throughout the interval as $\epsilon \to 0$ (cf. Ferguson (1975)). Such solutions will be well approximated within the interval by the n_1 dimensional span of the slow-mode matrix of (exact) solutions X_s, and the approximation will improve there as $\epsilon \to 0$. Near the endpoints, it is necessary to use the n_2 fast modes to provide appropriate boundary layer behavior.

(v) We note that the nonsingular transformation

$$x = \begin{bmatrix} I_{n_1} & -K \\ -L & I_{n_2} + LK \end{bmatrix} y, \tag{11}$$

where the $n_2 \times n_1$ matrix K satisfies the Liapunov equation

$$KB_2 - B_1K + A_{12} = 0, \tag{12}$$

converts (1) into the block-diagonal system

$$\dot{y} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} y, \tag{13}$$

whose slow and fast modes are completely decoupled. (Note that we could multiply the second equation of (13) through by the small parameter ϵ to achieve a system in the standard singularly perturbed form. The appropriate reduced (or outer) solution would be $Y_1(t) = e^{B_1 t} Y_1(0)$, $Y_2(t) = 0$.)

For the stable initial value problem for (1) (with $m_1 = n_2$), the exact slow-mode solution will be

$$x_s(t) = \begin{pmatrix} I_{n_1} \\ -L \end{pmatrix} e^{(A_{11}-A_{12}L)t} (I_{n_1} + KL \quad K)x(0) \tag{14}$$

where K is the unique solution of (12). If $K = 0$, we note that the first n_1 components of x will contain only slow modes. Otherwise, we obtain K by using successive approximations in the linear system

$$K = [(A_{11} - A_{12}L)K - A_{12}](A_{22} + LA_{12})^{-1}. \tag{15}$$

The scheme will converge rapidly because $\|B_1\|\|B_2^{-1}\| = O(\epsilon)$. It is important to realize that the slow-mode solution value $x_s(0)$ can differ considerably from

the prescribed value $x(0)$. This corresponds to the usual "boundary layer jump" of singular perturbations theory, showing that the reduced order (outer) solution (14) should only be used for $t > 0$. Indeed, the exact solution to the initial value problem is given by

$$x(t) = x_s(t) + x_f(t) \tag{16}$$

where the fast-mode (or boundary layer) solution x_f is

$$x_f(t) = \begin{pmatrix} -K \\ I_{n_2} + LK \end{pmatrix} e^{(A_{22}+LA_{12})t} (L \quad I_{n_2}) x(0). \tag{17}$$

Since $B_2 = A_{22} + LA_{12}$ has the large stable eigenvalues of F, x_f becomes asymptotically negligible as $\epsilon \to 0$ for $t > 0$ and the slow-mode solution x_s becomes a good approximation to $x(t)$ there.

(vi) Further details, including a justification for the algorithm and more general boundary layer structure, are contained in O'Malley and Anderson (1979) and in forthcoming work. Related discussions including examples of physical interest are contained in Chow and Kokotovic (1976a,b), Georgakis and Bauer (1978), Mattheij (1979), and Teneketzis and Sandell (1977).

(vii) The algorithm proposed has been applied to a number of physical systems with orders n up to 32. Rather good results were obtained in reducing the 16th order model of an F-100 turbofan engine to a 3rd order model, even though the ϵ involved was the not-so-small parameter 0.383.

The longitudinal dynamics of an F-8 aircraft can be modeled by a fourth-order linear system with slow eigenvalues $-0.0075 \pm i0.076$ and fast eigenvalues $-0.94 \pm i3.0$, corresponding to the small parameter 0.024. The physical variables of velocity variation and flight path angle exhibit primarily slow response, while the angle of attack and pitch rate are predominantly fast variables. This is illustrated in Figures 1 and 2 where both the exact solution $x(t)$ and the slow mode component $x_s(t)$ are plotted for typical slow and fast variables. We note that the fast mode is present in both variables, but that it quickly dies out so that $x(t)$ is well approximated by $x_s(t)$ in a few seconds.

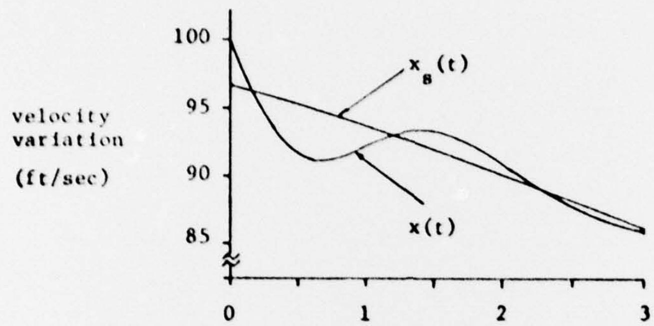

Figure 1. A slow physical variable.

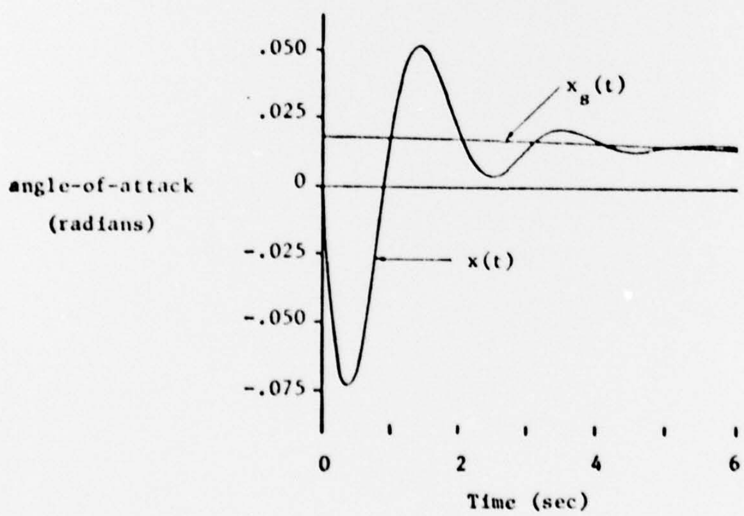

Figure 2. A fast physical variable.

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